Solutions to tutorial exercises for stochastic processes

T1. Denote by T the product topology on S. We will first show that the projection $\pi_v : S \to X$ given by $\pi_v(\eta) = \eta(v)$ is continuous. Let T_X denote the topology on X and let $A \in T_X$. For any $v \in V$ we have

$$\pi_v^{-1}(A) = \{\eta \in S \,:\, \eta(v) \in A\} = \prod_{w \neq v} X \times A \in T,$$

by the definition of the product topology. So π_v is continuous for all $v \in V$.

Now let T denote a topology on S such that π_v is continuous for all $v \in V$. So for all $A \in T_X$ we have

$$\pi_v^{-1}(A) = \prod_{w \neq v} X \times A \in T.$$

Suppose $B \subseteq S$ can be written as

$$B = \prod_{v \in V_1} B_v \times \prod_{v \in V_2} X,$$

where V_1 is finite, $V_1 \cup V_2 = S$ and $B_v \in T_X$. Then we can write

$$B = \bigcap_{v \in V_1} \left(\prod_{w \neq v} X \times B_v \right) \in T,$$

since V_1 is finite. It follows that T contains all sets included in the product topology.

T2. Let $A \in T_\rho$ and let $\eta \in A$ and $r > 0$ such that $\rho(\eta, \xi) < r$ implies $\xi \in A$. Since α is summable we can write

$$\sum_{v \in \mathbb{Z}^d} \alpha(v) = \sum_{v \in V_1} \alpha(v) + \sum_{v \in V_2} \alpha(v),$$

with V_1 finite and

$$\sum_{v \in V_2} \alpha(v) < r.$$

Consider the set

$$B_\eta = \prod_{v \in V_1} \{\eta(v)\} \times \prod_{v \in V_2} \{0, 1\}.$$

Then for all $\xi \in B_\eta$ it holds that $\rho(\eta, \xi) < r$, so that $B_\eta \subset A$. Furthermore we have $A = \bigcup_{\eta \in A} B_\eta$, so that A is in the product topology. So T_ρ is a subset of the product topology.

Now let B be in the base of the product topology:

$$B = \prod_{v \in V_1} B_v \times \prod_{v \in V_2} \{0, 1\},$$

with V_1 finite. Now let $\eta \in B$ and take $r = \min_{v \in V_1} \alpha(v)$. Then if $\rho(\eta, \xi) < r$ it follows that $\eta(v) = \xi(v)$ for all $v \in V_1$, so that $\xi \in B$. It follows that B is in T_ρ. Since T_ρ is a topology and the base of the product topology is contained in T_ρ it follows that the product topology is a subset of T_ρ.

T3. Since $\mathbb{Z}^d$ is countable, we can find a bijection $\nu : \mathbb{Z}^d \to \mathbb{N}$. Define $\alpha : \mathbb{Z}^d \to (0, \infty)$ by $\alpha(x) = 1/\nu(x)^2$. Then

$$\sum_{x \in \mathbb{Z}^d} \alpha(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Let ρ be the metric as defined in (T2). Using (T2), it remains to show that (η_n) converges pointwise if and only if it converges with respect to ρ.

$\Rightarrow$: Suppose η_n converges pointwise to η. For every $x \in \mathbb{Z}^d$ there exists $N_x \in \mathbb{N}$ such that for all $n > N_x$ we have $\eta_n(x) = \eta(x)$. Let $\varepsilon > 0$ and take M such that $\sum_{n=M}^{\infty} 1/n^2 < \varepsilon$. Let

$$N := \max_{x : \nu(x) < M} N_x.$$

It follows that for all $n > N$ we have

$$\rho(\eta_n, \eta) = \sum_{x \in \mathbb{Z}^d} \frac{1}{\nu(x)^2} |\eta_n(x) - \eta(x)| = \sum_{x : \nu(x) \geq M} \frac{1}{\nu(x)^2} < \varepsilon.$$

It follows that (η_n) converges with respect to ρ.

$\Leftarrow$: Suppose (η_n) converges with respect to ρ. Let $x \in \mathbb{Z}^d$. Take $\varepsilon = 1/\nu(x)^2$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\rho(\eta_n, \eta) = \sum_{x \in \mathbb{Z}^d} \frac{1}{\nu(x)^2} |\eta_n(x) - \eta(x)| < \varepsilon.$$

It follows that $\eta_n(x) = \eta(x)$ for all $n > N$, so that (η_n) converges pointwise.

T4. Consider the function $f : \{0, 1\}^V \to \{0, 1\}$ given by

$$f(\eta) = \mathbb{1}\big\{|x \in V \ : \ \eta(x) = 1| = \infty\big\}$$

. Then $\sup_\eta |f(\eta_x) - f(\eta)| = 0$ for all $x \in V$, so that

$$\sum_{x \in V} \sup_\eta |f(\eta_x) - f(\eta)| = 0.$$

However we can show that f is not continuous. Let $(x_k)_{k\in\mathbb{N}}$ be an enumeration of V and let η be such that $f(\eta) = 1$. Define the sequence

$$\eta_n(x) = \begin{cases} \eta(x) & \text{if } x = x_k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\eta_n(x) \to \eta(x)$ pointwise as $n \to \infty$, so also $\eta_n \to \eta$ by T3. However $f(\eta_n) = 0$ for all $n \in \mathbb{N}$.